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On asymmetric gravity–capillary solitary waves

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Symmetric gravity–capillary solitary waves with decaying oscillatory tails are known to bifurcate from infinitesimal periodic waves at the minimum value of the phase speed where the group velocity is equal to the phase speed. In the small-amplitude limit, these solitary waves may be interpreted as envelope solitons with stationary crests and are described by the nonlinear Schrödinger (NLS) equation to leading order. In line with this interpretation, it would appear that one may also construct asymmetric solitary waves by shifting the carrier oscillations relative to the envelope of a symmetric solitary wave. This possibility is examined here on the basis of the fifth-order Korteweg–de Vries (KdV) equation, a model for gravity–capillary waves on water of finite depth when the Bond number is close to $\frac{1}{3}$. Using techniques of exponential asymptotics beyond all orders of the NLS theory, it is shown that asymmetric solitary waves of the form suggested by the NLS theory in fact are not possible. On the other hand, an infinity of symmetric and asymmetric solitary-wave solution families comprising two or more NLS solitary wavepackets bifurcate at finite values of the amplitude parameter. The asymptotic results are consistent with numerical solutions of the fifth-order KdV equation. Moreover, the asymptotic theory suggests that such multi-packet gravity–capillary solitary waves also exist in the full water-wave problem near the minimum of the phase speed.

1. Introduction

The majority of previous theoretical studies of solitary water waves deal with symmetric waves. In the absence of surface tension, in fact, it has been established rigorously (Craig & Sternberg 1988, 1992) that gravity solitary waves travelling at supercritical speed (higher than the linear-long-wave speed) on water of finite depth can only be symmetric, consistent with the classical Korteweg–de Vries (KdV) theory (Whitham 1974, §13.11). In this paper, we wish to examine the possibility of asymmetric gravity–capillary solitary waves.

Our approach is motivated by the recent discovery of a new class of symmetric gravity–capillary solitary waves that feature tails with decaying oscillations. The origin of these waves is different from the familiar solitary waves of the KdV type: while KdV solitary waves bifurcate from infinitesimal long waves on water of finite depth, the new type of solitary waves bifurcate from infinitesimal periodic waves of finite wavenumber at the minimum value of the linear phase speed on water of finite or infinite depth; such a minimum is present only if the effects of surface tension are taken into account.

Longuet-Higgins (1989) first provided numerical evidence that gravity–capillary solitary waves exist on deep water. The waves he computed are all symmetric and actually are part of the so-called depression branch because the level of the free surface at the point of symmetry is below the free-surface level at infinity. Vanden-Broeck & Dias (1992) later computed two symmetric solitary-wave solution branches, one corresponding to depression and the other to elevation waves, and recognized the connection of these solitary waves with small-amplitude periodic wavetrains at the minimum of the gravity–capillary phase speed.

In related analytical work, Iooss & Kirchgässner (1990) showed rigorously that small-amplitude, symmetric, elevation and depression gravity–capillary solitary waves exist on water of finite depth and, more recently, Kirrmann & Iooss (1995) extended the proof to the case of infinite water depth. Using a similar theoretical approach, Dias & Iooss (1993) obtained analytical approximations of these solitary waves that compare favourably with the numerical results of Vanden-Broeck & Dias (1992) in the small-amplitude limit.

One may gain further insight into the origin of the new class of solitary waves by noting that, in the small-amplitude limit, they can be viewed as modulated wavepackets; this interpretation also proves useful in discussing asymmetric solitary waves. As is well known, slow modulations of weakly nonlinear, two-dimensional gravity–capillary wavepackets are governed by the nonlinear Schrödinger (NLS) equation to leading order (see, for example, Djordjeric & Redekopp 1977). The NLS equation admits symmetric envelope-soliton solutions with a 'sech' profile corresponding to locally confined wavepackets, but these packets are not waves of permanent form in general; the envelope travels with the group velocity while the carrier oscillations travel with the phase speed. At the minimum of the phase speed, however, the phase velocity is equal to the group velocity, and it is possible to construct envelope solitons with stationary crests (in the frame of the envelope) that correspond to solitary waves with oscillatory tails (Akylas 1993; Longuet-Higgins 1993). In agreement with Dias & Iooss (1993), two symmetric solution branches arise: elevation waves when the maximum of the envelope coincides with a crest of the carrier, and depression waves when the maximum of the envelope coincides with a trough of the carrier.

Based on the above interpretation, it would appear feasible to construct small-amplitude asymmetric solitary waves as well, by translating the crests of a symmetric solitary wave relative to its wave envelope. This is consistent with the NLS equation and, as it turns out, there is no contradiction to all orders of approximation in the corresponding two-scale perturbation expansion. On the other hand, it seems curious that asymmetric waves would be possible for any phase shift of the carrier oscillations. Moreover, as the rigorous existence proofs cited earlier apply to symmetric waves only, this heuristic reasoning does not guarantee that there exist exact asymmetric solitary-wave solutions of the water-wave problem; exponentially small corrections to the perturbation expansion could come into play, for example, precluding asymmetric solitary waves.

As a first step towards settling these issues, we shall focus on asymmetric solitary-wave solutions of the fifth-order KdV equation. This model equation can be formally derived from the full gravity–capillary water-wave problem for weakly nonlinear, long waves on water of finite depth when the Bond number is close to $\frac{1}{3}$ (see, for example, Hunter & Scheurle 1988). Even though neglecting viscosity cannot be justified under these flow conditions, the corresponding linear phase speed has a minimum at a finite wavenumber, and the fifth-order KdV equation is perhaps the simplest nonlinear dispersive equation that admits the class of solitary-wave solutions of interest here.

From previous work, the fifth-order KdV equation is known to have a rich structure of permanent-wave solutions and, in fact, some asymmetric solitary waves have been found numerically. Specifically, Zufiria (1987) was mostly interested in periodic waves. As he followed branches of symmetric periodic waves, he discovered symmetry-breaking bifurcations and, by increasing the wave period, he was the first to our knowledge to compute asymmetric solitary waves. In more recent work treating the (steady) fifth-order KdV equation as a dynamical system, Champneys & Toland (1993) and Buffoni, Champneys & Toland (1995) proved the existence of an infinity of homoclinic orbits corresponding to solitary waves. By following symmetric-solution branches numerically, they also found bifurcations into branches of asymmetric solitary waves.

In a similar vein to the present investigation, using a two-scale perturbation expansion near the minimum of the phase speed, Grimshaw, Malomed & Benilov (1994) constructed small-amplitude symmetric solitary-wavepacket solutions of the fifth-order KdV equation, analogous to the gravity–capillary waves of Dias & Iooss (1993). While not emphasized in their paper, the asymptotic theory of Grimshaw *et al.* (1994) suggests that asymmetric solitary wavepackets are also admissible solutions of the fifth-order KdV equation and bifurcate from infinitesimal periodic waves at the minimum of the phase speed, in accordance with the remarks made earlier. But this would seem to contradict the numerical findings of Zufiria (1987) and Buffoni *et al.* (1995) which suggest that asymmetric solitary waves bifurcate at finite amplitude.

We wish to understand the structure of solitary-wave solutions of the fifth-order KdV equation near the minimum of the phase speed. To reconcile the differences noted above between the asymptotic and numerical results, we carry the two-scale perturbation expansion of Grimshaw *et al.* (1994) beyond all orders in the small-amplitude parameter. The revised perturbation theory reveals that an NLS envelope soliton with stationary crests can in fact remain locally confined only when it is symmetric, in which case the peak of the envelope coincides with either a crest or a trough of the carrier wavetrain. Shifting the carrier oscillations relative to the envelope results in an asymmetric disturbance which fails to be a locally confined permanent-wave solution of the fifth-order KdV equation: growing (in space) oscillations of exponentially small magnitude inevitably appear on one side of the wavepacket. Owing to nonlinear effects, however, this growing tail evolves into a new wavepacket and, more interestingly, when the carrier oscillations have just the right phase, it is possible that the whole disturbance terminates, resulting in a solitary wave with two (asymmetric) packets. Otherwise, a third wavepacket forms and the process continues, locally confined solitary waves with three or more packets being possible for specific values of the phase of the carrier.

Based on our analysis using exponential asymptotics, only two solitary-wave solution branches bifurcate at infinitesimal amplitude; they correspond to single-packet, symmetric, elevation or depression waves, in agreement with the analytical and numerical results cited earlier. In addition, however, there exists a countable infinity of symmetric and asymmetric solitary-wave solution families with two packets. But, unlike single-packet solitary waves, each of these two-packet families bifurcates at a certain finite amplitude, and it appears that multi-packet solitary waves with any number of packets can be found as well at finite amplitude. This explains the 'plethora' of solitary waves found by Champneys & Toland (1993) and Buffoni *et al.* (1995). Moreover, the symmetric and asymmetric branches of two-packet solitary waves intersect in a bifurcation diagram, so it is appropriate to attribute the appearance of asymmetric solitary waves to a symmetry-breaking bifurcation (Zufiria 1987).

2. Preliminaries

For the purpose of seeking solitary-wave solutions, we shall work with the normalized steady version of the fifth-order KdV equation

$$-\,c\,u + 3u^2 + u_{xx} + u_{xxxx} = 0 \qquad (-\infty < x < \infty), \tag{2.1}$$

where c stands for the wave speed, and impose the boundary conditions

$$u \to 0 \qquad (x \to \pm\infty). \tag{2.2}$$

According to (2.1), the phase speed of infinitesimal periodic waves of wavenumber k is given by $c(k) = -k^2 + k^4$, and its minimum value $c_m = -\frac{1}{4}$ is attained at $k = k_m = 1/\sqrt{2}$. As already indicated, we are interested in localized wavepacket solutions near the minimum of the linear phase speed, so c will be taken to be close to c_m,

$$c = c_m - 2\epsilon^2, \tag{2.3}$$

assuming the parameter ϵ to be small ($0 < \epsilon \ll 1$).

In view of (2.2), the tails of a solitary-wave solution are governed by the linearized version of (2.1), which has four independent solutions:

$$\mathrm{e}^{\pm\epsilon\gamma x}\cos k_c x \qquad \text{and} \qquad \mathrm{e}^{\pm\epsilon\gamma x}\sin k_c x,$$

where

$$k_c = \tfrac{1}{2}\left(1 + \left(1 + 8\epsilon^2\right)^{1/2}\right)^{1/2} = k_m\left(1 + \epsilon^2 + \cdots\right), \tag{2.4a}$$

$$\gamma = \frac{1}{2\epsilon}\left(-1 + \left(1 + 8\epsilon^2\right)^{1/2}\right)^{1/2} = 1 - \epsilon^2 + \cdots\,. \tag{2.4b}$$

Enforcing conditions (2.2) rules out the two growing solutions, proportional to $\mathrm{e}^{-\epsilon\gamma x}$ for $x \to -\infty$ and to $\mathrm{e}^{\epsilon\gamma x}$ for $x \to \infty$, and the admissible far-field solutions can be combined into the following form:

$$u \sim a_\pm\,\mathrm{e}^{-\epsilon\gamma|x|}\cos(k_c x + \phi_\pm) \qquad (x \to \pm\infty). \tag{2.5}$$

Note, however, that (2.1) is invariant under translations in x. Therefore, we may specify the amplitude parameter a_- arbitrarily as this amounts to fixing the location of the origin $x = 0$; the phase constant ϕ_- then is the only free parameter far upstream ($x \to -\infty$). If now (2.1) is thought of as a propagation (marching) problem with known upstream conditions it would seem unlikely in general to eliminate the two growing solutions far downstream ($x \to \infty$) with only one free upstream parameter (ϕ_-), unless $u(x)$ is assumed to be symmetric with respect to some point. Nevertheless, as it turns out (see §7), it is still possible to find asymmetric solitary-wave solutions of (2.1) for specific values of ϕ_-.

3. Two-scale perturbation expansion

Based on a standard two-scale expansion, Grimshaw *et al.* (1994) constructed symmetric solitary-wavepacket solutions of the fifth-order KdV equation near the minimum of the phase speed. In this section, we shall briefly re-examine their asymptotic theory in connection with the possibility of asymmetric solitary wavepackets; this will bring out the need for a refined perturbation theory that takes into account exponentially small terms.

At the minimum of the linear phase speed, the group velocity is equal to the phase speed c_m of infinitesimal periodic waves of wavenumber k_m. This suggests seeking solitary-wave solutions of the fifth-order KdV equation near this minimum in the form of small-amplitude modulated wavepackets such that both the envelope and the carrier oscillations travel with speed c, slightly less than c_m in view of (2.3) and (2.5). Accordingly, u is taken to depend on x and the 'slow' variable $X = \epsilon x$, and (2.1) becomes

$$-c\,u + u_{xx} + u_{xxxx} + 3u^2 + 2\epsilon u_{xX} + 4\epsilon u_{xxxX} + \epsilon^2 u_{XX} + 6\epsilon^2 u_{xxXX} + 4\epsilon^3 u_{xXXX} + \epsilon^4 u_{XXXX} = 0. \tag{3.1}$$

The analysis proceeds by introducing the two-scale expansion

$$u = \epsilon\left\{A(X)\,\mathrm{e}^{\mathrm{i}k_c x} + \text{c.c.}\right\} + \epsilon^2\left\{A_2(X)\,\mathrm{e}^{2\mathrm{i}k_c x} + \text{c.c.} + A_0(X)\right\} + \cdots, \tag{3.2}$$

where c.c. stands for the complex conjugate. Upon substitution into (3.1), it is found that A_2 and A_0 are related to A by

$$A_2 = -\tfrac{4}{3}\,A^2 - \tfrac{128}{9}\,\mathrm{i}k_m\,\epsilon A A_X + O(\epsilon^2), \qquad A_0 = -24\,|A|^2 + O(\epsilon^2),$$

and, correct to $O(\epsilon)$, A is governed by

$$A - A_{XX} - 76|A|^2 A - 2\mathrm{i}k_m\epsilon(A_X - A_{XXX}) - \tfrac{128}{3}\,\mathrm{i}k_m\epsilon|A|^2 A_X + O(\epsilon^2) = 0. \tag{3.3}$$

To leading order, this is the NLS equation in steady form.

It is straightforward to show that (3.3) admits a locally confined (envelope-soliton) solution,

$$A(X) = S(X)\,\mathrm{e}^{\mathrm{i}\phi(X)},$$

where

$$S = \frac{1}{\sqrt{38}}\,\mathrm{sech}\,X + O(\epsilon^2), \qquad \phi = \phi_0 - \frac{187}{57\sqrt{2}}\,\epsilon\tanh X + O(\epsilon^2),$$

ϕ_0 being an arbitrary phase constant. Combining this expression for the wave envelope with the carrier oscillations, expansion (3.2) then yields, correct to $O(\epsilon^2)$,

$$u = \sqrt{\frac{2}{19}}\epsilon\cos(k_m x + \phi_0)\,\mathrm{sech}\,X + \epsilon^2\left\{\frac{187}{57\sqrt{19}}\,\sin(k_m x + \phi_0)\,\mathrm{sech}\,X\tanh X - \tfrac{4}{19}\left(3 + \tfrac{1}{3}\cos(2k_m x + 2\phi_0)\right)\,\mathrm{sech}^2 X\right\} + O(\epsilon^3). \tag{3.4}$$

This asymptotic solution describes a locally confined wavepacket with crests moving at the same speed as the envelope so, as a whole, (3.4) is a solitary wave. In particular, there are two symmetric (with respect to $x = 0$) solution branches when the phase constant $\phi_0 = 0$ or π, $\phi_0 = 0$ corresponding to elevation waves and $\phi_0 = \pi$ to depression waves, in agreement with Grimshaw *et al.* (1994).

Apart from these symmetric solitary-wave solutions, however, (3.4) suggests that asymmetric solitary waves are possible for all values of ϕ_0 other than 0 or π, and it would appear that these waves also bifurcate from periodic waves of infinitesimal amplitude. The numerical results of Zufiria (1987) and Buffoni *et al.* (1995), on the other hand, indicate that asymmetric solitary waves exist at finite amplitude only. Moreover, based on the asymptotic behaviours (2.5) at the solitary-wave tails, it seems unlikely that solitary waves would be possible for all values of ϕ_0; this would imply that one could eliminate the two growing solutions of the fifth-order KdV equation far downstream for any value of the upstream phase ϕ_-.

There is reason to suspect, therefore, that the predictions of the asymptotic expression (3.4) regarding asymmetric solitary waves are misleading. This discrepancy apparently cannot be remedied by carrying the two-scale expansion (3.2) to higher order: it can be checked that the wave envelope remains locally confined to all orders so one could still construct asymmetric solitary waves by shifting the carrier oscillations.

The cause of the difficulty must lie beyond all orders of the standard two-scale expansion (3.2), suggesting the need for a refined perturbation theory that accounts for exponentially small corrections. A similar approach, utilizing exponential asymptotics, has also proven useful in instances where KdV solitary waves develop oscillatory tails of exponentially small amplitude (see, for example, Pomeau, Ramani & Grammaticos 1988; Yang & Akylas (1995, 1996) and references given therein). The problem at hand is more complicated, however, because the solitary waves of interest here are wavepackets involving a carrier signal and its envelope which have different lengthscales.

The potential significance of exponentially small terms is brought out more clearly by working in the wavenumber domain. Taking the Fourier transform with respect to the slow variable X,

$$\widehat{u}(x,K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,X)\,\mathrm{e}^{-\mathrm{i}KX}\,\mathrm{d}X,$$

expansion (3.2) yields

$$\widehat{u} = \epsilon\,\mathrm{sech}\,\frac{\pi K}{2}\left\{\frac{1}{\sqrt{38}}\cos(k_m x+\phi_0) - \epsilon K\left[\mathrm{i}\frac{187}{114\sqrt{19}}\sin(k_m x+\phi_0) + \tfrac{2}{19}\left(3+\tfrac{1}{3}\cos(2k_m x+2\phi_0)\right)\coth\tfrac{1}{2}\pi K\right] + \cdots\right\}. \qquad (3.5)$$

It is interesting that this expansion, unlike its counterpart (3.4) in the physical domain, becomes disordered when $K = O(1/\epsilon)$. Based on previous experience (Akylas & Yang 1995), this non-uniformity suggests that $\widehat{u}(x,K)$ has singularities of exponentially small strength – the common factor sech $(\pi K/2)$ in (3.5) is exponentially small for $K = O(1/\epsilon)$ – close to the real K-axis. These singularities in turn reflect the presence of exponentially small oscillatory wave tails depending on X/ϵ in the physical domain. Since the envelope equation (3.3) was derived on the assumption that A depends on the slow variable X only, one has to return to the original equation (3.1) to compute these tails.

4. Revised perturbation theory

In line with the remarks above, we now revise the perturbation theory to account for exponentially small terms, beyond all orders of the standard two-scale expansion (3.2).

Following Akylas & Yang (1995), we find it convenient to work in the wavenumber domain. Taking the Fourier transform with respect to X, (3.1) is converted to an integral–differential equation for $\widehat{u}(x,K)$:

$$(-c-\epsilon^2K^2+\epsilon^4K^4)\widehat{u} + 2\mathrm{i}\epsilon K(1-2\epsilon^2K^2)\widehat{u}_x + (1-6\epsilon^2K^2)\widehat{u}_{xx} + 4\mathrm{i}\epsilon K\widehat{u}_{xxx} + \widehat{u}_{xxxx} + 3\widehat{u^2} = 0. \qquad (4.1)$$

The breakdown of expansion (3.5) when $\epsilon K = O(1)$ suggests the uniformly valid two-scale expression

$$\widehat{u} = \epsilon \operatorname{sech}\left(\tfrac{1}{2}\pi K\right) U(x,\kappa), \tag{4.2}$$

in terms of the scaled wavenumber variable $\kappa = \epsilon K$, with

$$U \sim \frac{1}{\sqrt{38}}\cos(k_m x + \phi_0) - \mathrm{i}\frac{187}{114\sqrt{19}}\,\kappa \sin(k_m x + \phi_0) \\ - \tfrac{2}{19}|\kappa|\left(3 + \tfrac{1}{3}\cos(2k_m x + 2\phi_0)\right) + \cdots \qquad (\kappa \to 0). \tag{4.3}$$

The goal of the ensuing analysis is to determine the behaviour of $U(x,\kappa)$ near its singularities in the κ-plane; upon inverting the Fourier transform, we shall then compute the exponentially small terms that arise in the physical domain from these singularities.

Substituting (4.2) into (4.1), $U(x,\kappa)$ satisfies

$$(-c - \kappa^2 + \kappa^4)U + 2\mathrm{i}\kappa(1-2\kappa^2)U_x + (1-6\kappa^2)U_{xx} + 4\mathrm{i}\kappa U_{xxx} \\ + U_{xxxx} + 3\cosh\frac{\pi\kappa}{2\epsilon}\int_{-\infty}^{\infty}\frac{U(x,\lambda)\,U(x,\kappa-\lambda)}{\cosh(\pi\lambda/2\epsilon)\,\cosh(\pi(\kappa-\lambda)/2\epsilon)}\,\mathrm{d}\lambda = 0. \tag{4.4}$$

The solution of (4.4) is posed as a Fourier series

$$U(x,\kappa) = \sum_{n=-\infty}^{\infty} \mathscr{A}_n(\kappa)\mathrm{e}^{\mathrm{i}n\theta_c}, \tag{4.5}$$

in terms of the total phase $\theta_c = k_c x + \phi_0$, where, in view of (4.3),

$$\mathscr{A}_0 \sim -\tfrac{6}{19}|\kappa| + \cdots \qquad (\kappa \to 0), \tag{4.6a}$$

$$\mathscr{A}_{\pm 1} \sim \frac{1}{2\sqrt{38}} \mp \frac{187}{228\sqrt{19}}\kappa + \cdots \qquad (\kappa \to 0), \tag{4.6b}$$

$$\mathscr{A}_{\pm 2} \sim -\tfrac{1}{57}|\kappa| + \cdots \qquad (\kappa \to 0), \tag{4.6c}$$

and $\mathscr{A}_{\pm n} = O(\kappa^{|n|-1})$ $(|n| > 2)$. Upon substitution of (4.5) into (4.4), the $\mathscr{A}_n$ $(n = 0, \pm 1, \pm 2, \ldots)$ are governed by the following system of coupled integral equations:

$$\left\{-c - \kappa^2 + \kappa^4 - 2nk_c\kappa(1-2\kappa^2) - n^2k_c^2(1-6\kappa^2) + 4n^3k_c^3\kappa + n^4k_c^4\right\}\mathscr{A}_n \\ + 3\cosh\frac{\pi\kappa}{2\epsilon}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathscr{A}_p(\lambda)\mathscr{A}_{n-p}(\kappa-\lambda)}{\cosh(\pi\lambda/2\epsilon)\,\cosh(\pi(\kappa-\lambda)/2\epsilon)}\,\mathrm{d}\lambda = 0 \\ (n = 0, \pm 1, \pm 2, \ldots). \tag{4.7}$$

No approximation has been made thus far, and the equation system (4.7) together with (4.5) and (4.2) is entirely equivalent to the original equation (3.1). However, in the limit $\epsilon \to 0$, the main contribution to the convolution integrals in (4.7) comes from the range $0 < \lambda < \kappa$ $(\kappa > 0)$, $\kappa < \lambda < 0$ $(\kappa < 0)$ as long as κ is not too close to $\pm k_m$ ($k_m - |\kappa| \gg O(\epsilon)$; see §5). Also, since u is real, $\mathscr{A}_n(-\kappa) = \mathscr{A}_{-n}(\kappa)$ on the real

κ-axis, and it suffices to consider $\mathscr{A}_n$ ($n > 0$) only. Therefore, taking into account (2.3) and (2.4a), (4.7) simplifies to

$$\{\kappa + (n+1)k_m\}^2\ \{\kappa + (n-1)k_m\}^2\ \mathscr{A}_n + 6\sum_{p=0}^{n} \operatorname{sgn}\kappa \int_0^{\kappa} \mathscr{A}_p(\lambda)\,\mathscr{A}_{n-p}(\kappa-\lambda)\,\mathrm{d}\lambda$$

$$+12\sum_{p=1}^{\infty} \operatorname{sgn}\kappa \int_0^{\kappa} \mathscr{A}_p(-\lambda)\,\mathscr{A}_{n+p}(\kappa-\lambda)\,\mathrm{d}\lambda = 0 \qquad (n \geqslant 0) \tag{4.8}$$

in the limit $\epsilon \to 0$.

Before proceeding to analyse equations (4.8), it is interesting to compare the approach taken here with the standard two-scale expansion outlined in §3. It is easy to check that equations (4.8) are consistent with (4.6) as $\kappa \to 0$, and the harmonics in the Fourier series (4.5) are ordered in this limit. The proposed expression (4.2) for $\widehat{u}(x, K)$, therefore, merges smoothly in the matching region $1 \ll |K| \ll 1/\epsilon$ with expansion (3.5) which derives from the standard two-scale expansion (3.2). On the other hand, (4.2), combined with (4.5) and (4.8), remains valid for $\kappa = O(1)$ where, as will be seen, the Fourier coefficients $\mathscr{A}_n(\kappa)$ have singularities; these singularities amount to exponentially small terms in the physical domain, beyond all orders of the two-scale expansion (3.2).

Specifically, the coefficient of $\mathscr{A}_n$ in (4.8) vanishes when $\kappa = -(n \pm 1)k_m$ ($n \geqslant 0$) and $\mathscr{A}_n(\kappa)$ would appear to be singular there. Out of these possible singularities, those closest to the origin are located at $\kappa = \pm k_m$ (the origin $\kappa = 0$ is a regular point in view of (4.6)) and make the dominant contribution.

5. Behaviour near the singularities

Attention is now focused on the local behaviour of $\mathscr{A}_n(\kappa)$ ($n \geqslant 0$) near $\kappa = -k_m$. As expected, $\mathscr{A}_0(\kappa)$ and $\mathscr{A}_2(\kappa)$ are the most singular since their coefficients in (4.8) vanish when $\kappa = -k_m$. However, through the convolution integrals in (4.8), the singularities of $\mathscr{A}_1(\kappa)$ at $\kappa = \pm k_m$ and of $\mathscr{A}_3(\kappa)$ at $\kappa = -k_m$ also participate in the dominant balance near $\kappa = -k_m$ (see the Appendix for details). It turns out that

$$\mathscr{A}_0 \sim \frac{C}{(\kappa + k_m)^4}, \quad \mathscr{A}_2 \sim \frac{C}{(\kappa + k_m)^4} \qquad (\kappa \to -k_m), \tag{5.1a}$$

$$\mathscr{A}_1 \sim -\frac{16}{\sqrt{38}}\,\frac{C}{(\kappa + k_m)^3}, \quad \mathscr{A}_3 \sim -\frac{8}{9\sqrt{38}}\,\frac{C}{(\kappa + k_m)^3} \qquad (\kappa \to -k_m), \tag{5.1b}$$

the rest of the $\mathscr{A}_n$ being less singular. The constant C above is determined by solving (4.8) subject to (4.6) numerically. Following the procedure described in the Appendix, we compute $C = -0.011$. It is worth noting that C depends on all the coefficients $\mathscr{A}_n(\kappa)$ of the Fourier expansion (4.5), so it contains information beyond all orders of the standard two-scale expansion.

The asymptotic behaviours (5.1) were deduced on the basis of the approximate equation system (4.8) for the Fourier coefficients $\mathscr{A}_n(\kappa)$. Returning to the exact system (4.7), however, note that, in view of (4.6) and (5.1), it is not permissible to approximate the convolution integrals in (4.7) by those in (4.8) when κ is very close to $\pm k_m$, $k_m - |\kappa| \leqslant O(\epsilon)$. Therefore, (5.1) are expected to break down in the immediate vicinity of $\kappa = -k_m$.

To handle this complication, following a matched-asymptotics procedure in terms of the 'inner' wavenumber variable $\sigma = (\kappa + k_m)/\epsilon$, we shall determine local solutions for $\mathscr{A}_0$ and $\mathscr{A}_2$, valid when $\sigma = O(1)$. Specifically, the asymptotic behaviours (5.1) suggest the rescalings

$$\mathscr{A}_0 = \frac{1}{\epsilon^4}\,\Phi_0(\sigma), \quad \mathscr{A}_2 = \frac{1}{\epsilon^4}\,\Phi_2(\sigma), \tag{5.2a}$$

$$\mathscr{A}_1 = \frac{1}{\epsilon^3}\,\Phi_1(\sigma), \quad \mathscr{A}_3 = \frac{1}{\epsilon^3}\,\Phi_3(\sigma). \tag{5.2b}$$

Substituting (5.2) into (4.7) taking into account (4.6), to leading order, it is found that $\Phi_0 = \Phi_2 \equiv \Phi$ and

$$\Phi_1(\sigma) = -\frac{24}{\sqrt{38}}\int_{-\infty}^{\infty} \mathrm{d}l\, \mathrm{e}^{-\pi l/2}\,\mathrm{sech}\,(\tfrac{1}{2}\pi l)\,\Phi(\sigma - l), \qquad \Phi_3(\sigma) = \tfrac{1}{18}\,\Phi_1(\sigma),$$

where $\Phi(\sigma)$ satisfies the linear integral equation

$$(\sigma^2+1)\,\Phi(\sigma) - \int_{-\infty}^{\infty} \mathrm{d}l\, l\, \mathrm{e}^{-\pi l/2}\,\mathrm{cosech}\,(\tfrac{1}{2}\pi l)\,\Phi(\sigma - l)$$

$$- \int_{-\infty}^{\infty} \mathrm{d}l\, \mathrm{e}^{-\pi l/2}\,\mathrm{sech}\,(\tfrac{1}{2}\pi l) \int_{-\infty}^{\infty} \mathrm{d}l_1\, \mathrm{e}^{-\pi l_1/2}\,\mathrm{sech}\,(\tfrac{1}{2}\pi l_1)\,\Phi(\sigma - l - l_1) = 0. \tag{5.3}$$

Furthermore, to be consistent with (5.1), the matching condition

$$\Phi \sim \frac{C}{\sigma^4} \qquad (\sigma \to \infty) \tag{5.4}$$

is imposed.

Guided by previous experience (Akylas & Yang 1995), we pose the solution of (5.3) in the form

$$\Phi(\sigma) = \int_{\mathscr{L}} \mathrm{e}^{-\eta\sigma}\,\Psi(\eta)\,\mathrm{d}\eta, \tag{5.5}$$

where the contour $\mathscr{L}$ extends from $\eta = 0$ to ∞ with $\mathrm{Re}\,\eta\sigma > 0$. The integral equation (5.3) then formally transforms into a second-order differential equation for $\Psi(\eta)$:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\eta^2} + \left(1 - \frac{6}{\sin^2\eta}\right)\Psi = 0, \tag{5.6}$$

where, in view of (5.4), $\Psi(\eta) \sim \frac{1}{6}C\,\eta^3$ $(\eta \to 0)$.

It can be readily verified that $\cos\eta/\sin^2\eta$ is a solution of (5.6), although not consistent with the matching condition as $\eta \to 0$. Making use of this particular solution, the desired solution is found to be

$$\Psi(\eta) = \tfrac{5}{12}C\left(\frac{2}{\sin\eta} + \frac{\cos^2\eta}{\sin\eta} - \frac{3\eta\cos\eta}{\sin^2\eta}\right).$$

Returning to (5.5), one then has

$$(\sigma^2+1)\,\Phi(\sigma) = 6\int_{\mathscr{L}} \frac{\mathrm{e}^{-\eta\sigma}}{\sin^2\eta}\,\Psi(\eta)\,\mathrm{d}\eta,$$

and rotating the integration path $\mathscr{L}$ to the imaginary η-axis gives

$$(\sigma^2+1)\,\Phi(\sigma) \sim -\tfrac{5}{12}C \qquad (\sigma \to \pm\mathrm{i}).$$

Hence, $\Phi(\sigma)$ has simple-pole singularities at $\sigma = \pm\mathrm{i}$:

$$\Phi(\sigma) \sim \mp\mathrm{i}\frac{D}{\sigma\mp\mathrm{i}} \qquad (\sigma \to \pm\mathrm{i}), \tag{5.7}$$

where $D = -\frac{5}{24}C = 0.0023$.

Combining (5.7) with (4.2), (4.5) and (5.2), the simple-pole singularities of $\Phi(\sigma)$ at $\sigma = \pm\mathrm{i}$ translate into simple-pole singularities of $\widehat{u}(x,K)$ at $K = -(k_m/\epsilon) \pm \mathrm{i}$:

$$\widehat{u} \sim \frac{2D}{\epsilon^3}\exp\left(-\frac{\pi k_m}{2\epsilon}\right)\frac{1+\mathrm{e}^{2\mathrm{i}\theta_m}}{K+k_m/\epsilon\mp\mathrm{i}} \qquad \left(K \to -k_m/\epsilon \pm \mathrm{i}\right), \tag{5.8a}$$

where $\theta_m = k_m x + \phi_0$. Furthermore, since $u(x,X)$ is real, there is an additional pair of simple-pole singularities at $K = (k_m/\epsilon) \pm \mathrm{i}$:

$$\widehat{u} \sim -\frac{2D}{\epsilon^3}\exp\left(-\frac{\pi k_m}{2\epsilon}\right)\frac{1+\mathrm{e}^{-2\mathrm{i}\theta_m}}{K-k_m/\epsilon\mp\mathrm{i}} \qquad \left(K \to k_m/\epsilon \pm \mathrm{i}\right). \tag{5.8b}$$

As expected, the residues of these singularities in the wavenumber domain are exponentially small as $\epsilon \to 0$.

6. Single-packet solitary waves

Returning now to the physical domain, recall that the upstream amplitude parameter a_- in the asymptotic expressions (2.5) for the solitary-wave tails can be specified arbitrarily as this fixes the origin $x = 0$. For convenience, we shall choose the value of a_- as predicted by the straightforward expansion (3.4).

Accordingly, in taking the inverse Fourier transform

$$u(x,X) = \int_{\mathscr{C}} \widehat{u}(x,K)\,\mathrm{e}^{\mathrm{i}KX}\,\mathrm{d}K, \tag{6.1}$$

the contour $\mathscr{C}$ is indented to pass below the poles of $\widehat{u}(x,K)$ at $K = \pm(k_m/\epsilon) - \mathrm{i}$. With this choice of $\mathscr{C}$, the singularities in (5.8) play no role for $X < 0$ and, in view of (4.2), the dominant contribution to the wave profile far upstream comes from the pole of sech $(\pi K/2)$ at $K = -\mathrm{i}$:

$$u \sim \sqrt{\tfrac{8}{19}}\,\epsilon\,\mathrm{e}^{\epsilon x}\cos(k_m x + \phi_0) \qquad (x \to -\infty), \tag{6.2a}$$

consistent with the two-scale expansion (3.4) to leading order. The downstream disturbance, on the other hand, apart from a decaying oscillatory tail analogous to (6.2a), in general features oscillatory waves of exponentially small but growing (in space) amplitude owing to the contribution of the singularities in (5.8):

$$u \sim -\frac{16\pi D}{\epsilon^3}\exp\left(-\frac{\pi k_m}{2\epsilon}\right)\sin\phi_0\,\mathrm{e}^{\epsilon x}\cos(k_m x + \phi_0) + \sqrt{\tfrac{8}{19}}\,\epsilon\,\mathrm{e}^{-\epsilon x}\cos(k_m x + \phi_0) \qquad (x \to \infty). \tag{6.2b}$$

Based on (6.2), it is now clear that only symmetric small-amplitude wavepackets, for which $\phi_0 = 0$ or π so $\sin\phi_0 = 0$, remain locally confined and hence correspond to genuine solitary-wave solutions; shifting the carrier oscillations of a symmetric solitary wavepacket relative to its envelope results in an asymmetric disturbance which, however, fails to be a locally confined steady solution of the fifth-order KdV equation by exponentially small terms.

According to the revised perturbation theory, therefore, only two solitary-wave solution branches bifurcate from periodic waves of infinitesimal amplitude at the minimum of the phase speed; they correspond to symmetric elevation ($\phi_0 = 0$) or depression ($\phi_0 = \pi$) waves, consistent with the existence proofs for gravity–capillary waves (Iooss & Kirchgässner 1990; Kirrmann & Iooss 1995). As already noted, however, Champneys & Toland (1993) and Buffoni *et al.* (1995) found a plethora of other solitary waves close to the minimum of the phase speed that bifurcate at small but finite amplitude, and one wonders whether the revised perturbation theory can capture these solution branches.

The key to addressing this question is to note that, according to the asymptotic result (6.2*b*), u grows exponentially far downstream when $\phi_0 \neq 0, \pi$; owing to nonlinear effects a second wavepacket then is expected to arise, and it is interesting to ask whether it is possible that the wave disturbance decays to zero after two (or more) wavepackets have formed, resulting in a solitary wave with more than one packet. This possibility is examined in the next section.

7. Two-packet solitary waves

The construction of two-packet solitary waves involves piecing together two wavepackets. For this purpose, we shall rely on the asymptotic results (6.2) to ensure that the tails of the individual packets match smoothly.

In preparation for this matching, without carrying the exponential asymptotics to higher order, we shall refine expressions (6.2) heuristically. According to (2.4), we replace the carrier wavenumber k_m with the more accurate value k_c and use $\epsilon\gamma$ instead of ϵ for the decay and growth rate of the tails, so that far upstream

$$u \sim \sqrt{\tfrac{8}{19}}\,\epsilon\, \mathrm{e}^{\epsilon\gamma x} \cos(k_c x + \phi_-) \qquad (x \to -\infty) \tag{7.1a}$$

for some ϕ_-. Furthermore, taking into account the $O(\epsilon^2)$ corrections in the two-scale expansion (3.4), symmetric solitary waves about $x = 0$ ($\phi_0 = 0, \pi$) require that $\widetilde{\phi}_- \equiv \phi_- - \frac{187}{57\sqrt{2}}\,\epsilon = 0, \pi$ in which case no growing oscillations are expected downstream. Hence, (6.2*b*) is replaced with

$$u \sim -\frac{16\pi D}{\epsilon^3} \exp\left(-\frac{\pi k_c}{2\epsilon}\right) \sin\widetilde{\phi}_-\, \mathrm{e}^{\epsilon\gamma x} \cos(k_c x + \phi_-) + \sqrt{\tfrac{8}{19}}\,\epsilon\, \mathrm{e}^{-\epsilon\gamma x} \cos(k_c x + \phi_-) \qquad (x \gg 1). \tag{7.1b}$$

One also expects $O(\epsilon)$ phase shifts of the carrier oscillations downstream but it turns out that these corrections play no role to the order of approximation considered in the analysis below. Based on (7.1), we now proceed to construct solitary waves consisting of two wavepackets.

In addition to being invariant under a translation of the origin, the steady fifth-order KdV equation (2.1) preserves its form under the change of coordinate $x \to -x$. It is therefore legitimate to read (7.1) from the opposite direction and to translate the whole solution by a distance L. Specifically, if, for some ϕ_+ with $\widetilde{\phi}_+ \equiv \phi_+ + \frac{187}{57\sqrt{2}}\,\epsilon$,

$$u \sim \frac{16\pi D}{\epsilon^3} \exp\left(-\frac{\pi k_c}{2\epsilon}\right) \sin\widetilde{\phi}_+\, \mathrm{e}^{-\epsilon\gamma(x-L)} \cos(k_c x - k_c L + \phi_+)$$

$$+\sqrt{\tfrac{8}{19}}\,\epsilon\, \mathrm{e}^{\epsilon\gamma(x-L)} \cos(k_c x - k_c L + \phi_+) \qquad (1 \ll x \ll L) \qquad (7.2a)$$

on the left-hand tail of a packet centred at $x = L$, it follows that

$$u \sim \sqrt{\tfrac{8}{19}}\,\epsilon\, \mathrm{e}^{-\epsilon\gamma(x-L)} \cos(k_c x - k_c L + \phi_+) \qquad (x \gg L) \qquad (7.2b)$$

on the right-hand tail. Hence, two-packet solitary waves can be constructed if there exist some L, ϕ_- and ϕ_+ such that (7.1*b*) perfectly matches (7.2*a*). The parameter L in (7.2) then measures the distance between the centres of the two packets that make up the solitary wave. Without any loss of generality, we shall take $-\pi < \phi_\pm < \pi$ in the discussion below.

Perfect matching of (7.1*b*) with (7.2*a*) requires that

$$|\sin\widetilde{\phi}_-| = |\sin\widetilde{\phi}_+| = \frac{\epsilon^4}{4\sqrt{38}\pi D} \exp\left(\frac{\pi k_c}{2\epsilon} - \epsilon\gamma L\right), \qquad (7.3a)$$

$$\phi_+ + \chi_+ = k_c L + \phi_- + 2m_1\pi, \qquad (7.3b)$$

$$\phi_+ - \chi_- = k_c L + \phi_- + 2m_2\pi, \qquad (7.3c)$$

where $\chi_+ = 0(\pi)$ if $0 < \widetilde{\phi}_+ < \pi$ $(-\pi < \widetilde{\phi}_+ < 0)$, $\chi_- = \pi(0)$ if $0 < \widetilde{\phi}_- < \pi$ $(-\pi < \widetilde{\phi}_- < 0)$, and m_1, m_2 are integers. Subtracting (7.3*c*) from (7.3*b*) yields

$$\chi_+ + \chi_- = 2(m_1 - m_2)\pi.$$

Therefore, the phase constants $\widetilde{\phi}_+$ and $\widetilde{\phi}_-$ must have opposite signs, and condition (7.3*a*) can be satisfied in two ways: (i) $\widetilde{\phi}_+ = -\widetilde{\phi}_-$; (ii) $\widetilde{\phi}_+ = \widetilde{\phi}_- + \pi$ (if $-\pi < \widetilde{\phi}_- < 0$) or $\widetilde{\phi}_+ = \widetilde{\phi}_- - \pi$ (if $0 < \widetilde{\phi}_- < \pi$).

In view of (7.1*a*) and (7.2*b*), case (i) corresponds to solitary waves that are symmetric with respect to $x = \frac{1}{2}L$ and consist of two identical asymmetric (about their own centres) wavepackets. Also, in this case, (7.3*b*) gives

$$k_c L = m\pi - 2\phi_- + (-1)^m \pi, \qquad (7.4)$$

where m is an integer, even and odd values of m corresponding to positive and negative values of the phase constant ϕ_-, respectively. Equations (7.3*a*) and (7.4) can be combined to yield an algebraic equation for ϕ_-:

$$|\sin\widetilde{\phi}_-| = \frac{\epsilon^4}{4\sqrt{38}\pi D} \exp\left(\frac{\pi k_c}{2\epsilon} - \frac{\epsilon\gamma\,(m + (-1)^m)\,\pi}{k_c} + \frac{2\epsilon\gamma\phi_-}{k_c}\right). \qquad (7.5)$$

On the other hand, case (ii) corresponds to asymmetric waves in general and the distance L, correct to $O(1)$, is given by

$$k_c L = m\pi, \qquad (7.6)$$

m being an integer that labels the solution branches as in case (i) above. Upon substitution of (7.6) into (7.3*a*), it is found that the phase constant ϕ_- satisfies

$$|\sin\widetilde{\phi}_-| = \frac{\epsilon^4}{4\sqrt{38}\pi D} \exp\left(\frac{\pi k_c}{2\epsilon} - \frac{\epsilon\gamma m\pi}{k_c}\right). \qquad (7.7)$$

It is clear from (7.4) and (7.6) that the integer m controls the number of carrier wavelengths between the centres of the two packets that make up a solitary wave. The present asymptotic theory is expected to provide a good approximation to these waves when m is large, so that the two packets are well separated and the connection formulae (7.1) and (7.2) are valid. We also remark that the ignored $O(\epsilon)$ phase shifts of the carrier oscillations in (7.1b) and (7.2a) result in $O(\epsilon)$ corrections to L in (7.4) and (7.6) that, in turn, cause $O(\epsilon^2)$ changes in the arguments of the exponentials in equations (7.5) and (7.7) for ϕ_-. These corrections, therefore, impact ϕ_- to $O(\epsilon^2)$ and are insignificant to the order considered here.

Since $|\sin\widetilde{\phi}_-|$ cannot exceed 1, equations (7.5) and (7.7) have solutions for a fixed value of m, only if ϵ is above a certain threshold value which decreases as m is increased. Hence, all solution branches of symmetric and asymmetric two-packet solitary waves bifurcate at finite amplitude, consistent with the results of Champneys & Toland (1993) and Buffoni *et al.* (1995). Also, for a fixed value of ϵ, there exists an infinite number of such solution branches corresponding to values of m greater than a certain minimum.

From (7.5) and (7.7), symmetric and asymmetric solution branches corresponding to the same m bifurcate at slightly different values of the amplitude parameter ϵ. Symmetric branches intersect asymmetric branches, however, at the minimum amplitude of asymmetric waves where $\widetilde{\phi}_\pm = \pm\frac{1}{2}\pi$ or $\mp\frac{1}{2}\pi$ according to (7.7). It is therefore possible to encounter asymmetric solitary-wave branches bifurcating from symmetric ones as in Zufiria (1987) and Buffoni *et al.* (1995).

It is believed that the procedure outlined above for constructing two-packet solitary waves can be generalized to discuss solitary waves with any number of packets but the details are complicated by the presence of more than one matching region. In particular, the interior wavepacket(s) of a multi-packet solitary wave feature growing and decaying oscillations on both their left-hand and right-hand tails, so the connection formulae (7.1) and (7.2) need to be modified before matching these tails. In analogy with two-packet solitary waves, it is expected that solution families with n packets exist at finite amplitude only and are characterized by $n-1$ integers that control the number of carrier wavelengths between the maximum peaks of the packets.

We now turn to a discussion of numerical solutions to confirm the predictions of the asymptotic theory for two-packet solitary waves.

8. Numerical results

As there are exponentially growing solutions in the far field that could give rise to numerical instability, it is not convenient to solve the fifth-order KdV equation numerically as a one-way marching problem, so we follow a shooting procedure instead. Truncating the computational domain at some large distance $x = \pm x_\infty$, equation (2.1) is integrated from both $x = -x_\infty$ and $x = x_\infty$ to a matching point in between, $x = 0$ say, using a fourth-order Runge–Kutta method and starting conditions of the form (2.5). As remarked earlier, a_- can be specified arbitrarily; we choose $a_- = \sqrt{8/19}\,\epsilon$, consistent with (7.1$a$). We thus have three free parameters ($\phi_\pm$ and a_+) while there are four quantities (u, u_x, u_{xx} and u_{xxx}) to be matched at $x = 0$. Naturally, mismatches are present at $x = 0$ in general so it involves some searching in the parameter space to find a smooth solution. After a solution is found, the whole branch on which this solution lies can be traced by slowly varying the parameters.

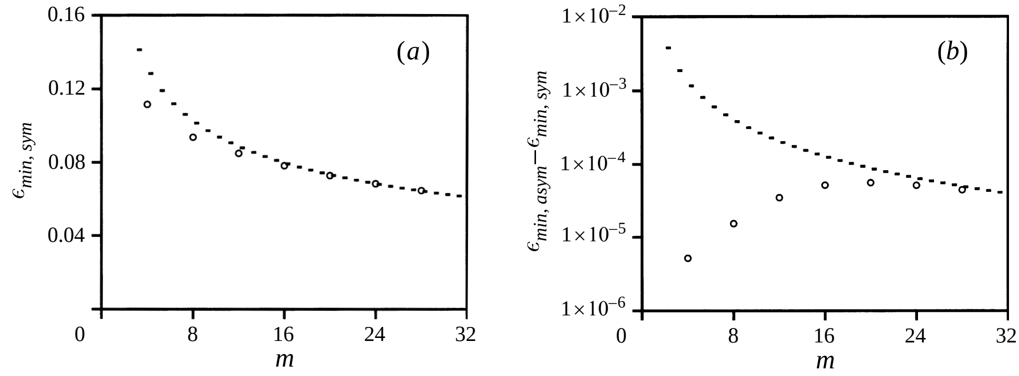

FIGURE 1. Comparison, for various values of m, of asymptotic results (- - - -) against numerical results (○) for the minimum amplitudes of two-packet solitary-wave solution families. (a) Minimum amplitude of symmetric waves; (b) difference between the minimum amplitudes of asymmetric and symmetric waves corresponding to the same value of m.

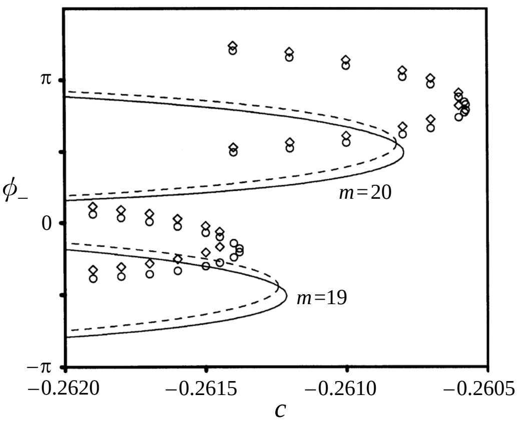

FIGURE 2. Solution branches of symmetric and asymmetric two-packet solitary waves corresponding to $m = 19$ and $m = 20$ in a bifurcation diagram where the upstream phase constant ϕ_- is plotted against the wave speed $c = -\frac{1}{4}-2\epsilon^2$. Both the predictions of the asymptotic theory (——, symmetric waves; - - - -, asymmetric waves) and numerical results (○ , symmetric waves; ◇, asymmetric waves) are shown.

A systematic study of two-packet solitary-wave solutions was carried out. Figure 1(a) shows the numerically computed minimum value of ϵ at which symmetric solution branches bifurcate for various values of m. The predictions of the asymptotic theory as obtained from (7.5) are also plotted for comparison. As expected, the asymptotic theory is valid for large m, when the two packets are well separated and the minimum amplitude is small, but there is reasonable agreement even when m is not all that large. The difference between the minimum amplitudes of symmetric and asymmetric waves corresponding to the same value of m is shown in figure 1(b) for various m; the difference is very small but is never zero. While there is good agreement for large m, the asymptotic and numerical results exhibit opposite trends as m decreases and, not unexpectedly, the asymptotic theory fails when m is small.

Figure 2 shows four solution branches of symmetric and asymmetric solitary waves corresponding to $m = 19$ and $m = 20$, in a bifurcation diagram where the upstream phase constant ϕ_- is plotted against the wave speed $c = -\frac{1}{4}-2\epsilon^2$. For these moderate

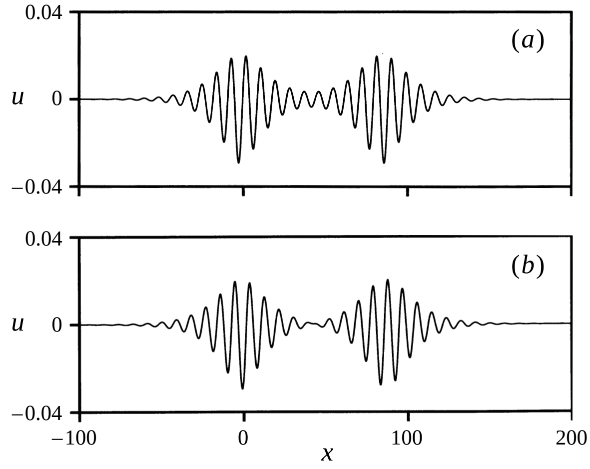

FIGURE 3. Profiles of numerically computed two-packet solitary waves. (*a*) Symmetric wave corresponding to $m = 19$, $c = -0.2615$, $\phi_- = -0.303\pi$; (*b*) asymmetric wave corresponding to $m = 20$, $c = -0.2614$, $\phi_- = 1.24\pi$.

values of m, the predictions of the asymptotic theory (equations (7.5) and (7.7)) agree qualitatively with the numerical computations. Each solution branch bifurcates at a finite value of ϵ where a turning point occurs and the wave speed attains its maximum value. For values of c less than the maximum value, there are four solitary-wave solutions, two symmetric and two asymmetric, corresponding to the same value of m. Also, it is interesting to note that the asymmetric branches intersect the symmetric branches near their turning points, consistent with the numerical results of Buffoni *et al.* (1995). Figure 3 displays the wave profiles of two solitary waves belonging to these branches. In both examples, as expected, the two packets that make up the solitary waves are asymmetric about their own centres. In figure 3(*a*), these packets are identical and form a symmetric wave as a whole while, in figure 3(*b*), they are different but still match smoothly to form an asymmetric wave.

9. Discussion

Using the fifth-order KdV equation as a simple model, we studied small-amplitude gravity–capillary solitary waves travelling with speed slightly below the minimum linear phase speed. In addition to the two previously known symmetric solution branches, that bifurcate from infinitesimal periodic waves at the minimum phase speed and correspond to NLS envelope solitons with stationary crests, there is an infinite number of symmetric and asymmetric solitary waves featuring more than one such wavepacket. These multi-packet solution branches bifurcate at finite amplitude, however, and cannot be captured by the NLS theory; in piecing together the individual wavepackets that make up a multi-packet solitary wave, it is necessary to take into account exponentially small terms, beyond all orders of the standard two-scale expansion on which the NLS equation is based.

Even though the fifth-order KdV equation is of limited validity as a model for gravity–capillary waves, the general picture that emerges from the asymptotic theory regarding small-amplitude solitary waves is expected to persist in the full water-wave problem close to the minimum of the phase speed. The only essential difference is the value of the constant D that enters in the connection formulae (7.1) and (7.2);

as it involves all nonlinear and dispersive terms, computing D in the full water-wave problem would not be an easy task. Apart from water waves, the conclusions of the asymptotic theory should also hold close to a phase-speed extremum in other dispersive wave systems. One example is interfacial waves for which Benjamin (1992) has found symmetric single-packet solitary waves using a model equation.

The asymptotic theory is valid in the small-amplitude limit when the packets that make up a solitary wave are well separated. This brings up the question as to what happens to the branches of multi-packet solitary waves when the amplitude parameter ϵ is increased and the individual packets lose their identity as they merge together. Although we are not prepared to address this question, the recent numerical results of Dias, Menasce & Vanden-Broeck (1996) seem to provide a clue: following the symmetric branch of single-packet elevation solitary waves in deep water away from its bifurcation point at infinitesimal amplitude, Dias *et al.* (1996) encountered turning points where new symmetric wave profiles, with more than one hump separated by smaller oscillations, bifurcate. It would appear that these multi-modal waves are related to symmetric multi-packet solitary waves. Their precise connection needs further investigation, however, and the fate of asymmetric-solution branches remains an entirely open question.

In recent laboratory observations of gravity–capillary waves in a wind-wave tank, Zhang (1995) found remarkably good agreement between some of the observed wave profiles and large-amplitude, symmetric, gravity–capillary solitary waves of depression on deep water as computed by Longuet-Higgins (1989). This suggests that the potential flow assumption is a reasonable approximation, and makes one wonder whether multi-packet solitary waves play a part as well.

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Appendix. Details of the asymptotic analysis

Here we give further details of the dominant-balance analysis that leads to the asymptotic expressions (5.1) and outline the numerical procedure for determining the constant C that appears in (5.1).

As remarked in §5, the coefficient of $\mathscr{A}_n(\kappa)$ in (4.8) vanishes when $\kappa = -(n \pm 1)\, k_m$, and all harmonics $\mathscr{A}_n(\kappa)$ $(n \geqslant 0)$ are expected to be singular there owing to their nonlinear coupling via the convolution integrals in (4.8). Since the origin $\kappa = 0$ actually is a regular point in view of (4.6), the singularities closest to the origin are located at $\kappa = \pm k_m$ and make the dominant contribution.

Attention is now focused on the singularities of $\mathscr{A}_n(\kappa)$ $(n \geqslant 0)$ at $\kappa = -k_m$; since their coefficients in (4.8) vanish when $\kappa = -k_m$, $\mathscr{A}_0(\kappa)$ and $\mathscr{A}_2(\kappa)$ are expected to be the most singular there. Note, however, that through the last sum of convolution integrals in (4.8), the singularities of other $\mathscr{A}_n(\kappa)$ $(n \geqslant 0)$ at $\kappa = k_m$ also participate in the dominant balance as $\kappa \to -k_m$.

Specifically, assuming that

$$\mathscr{A}_0 \sim \frac{C^{(0)}}{(k_m \mp \kappa)^N} \quad (\kappa \to \pm k_m), \qquad \mathscr{A}_2 \sim \frac{C^{(2)}}{(k_m + \kappa)^N} \quad (\kappa \to -k_m), \tag{A 1a}$$

where N is to be determined, it follows from (4.8), taking into account (4.6), that

$$\mathscr{A}_1 \sim \frac{C^{(\pm 1)}}{(k_m \mp \kappa)^{N-1}} \quad (\kappa \to \pm k_m), \qquad \mathscr{A}_3 \sim \frac{C^{(3)}}{(k_m + \kappa)^{N-1}} \quad (\kappa \to -k_m). \tag{A 1b}$$

The constants $C^{(m)}$ ($m = 0, \pm 1, 2, 3$) are related by

$$C^{(0)} = \frac{2}{19}\,\frac{18C^{(0)} + C^{(2)}}{(N-1)(N-2)} - \frac{3}{\sqrt{38}}\,\frac{C^{(1)} + C^{(-1)}}{N-2}, \tag{A 2}$$

$$C^{(2)} = \frac{2}{19}\,\frac{18C^{(2)} + C^{(0)}}{(N-1)(N-2)} - \frac{3}{\sqrt{38}}\,\frac{C^{(-1)} + C^{(3)}}{N-2}, \tag{A 3}$$

$$C^{(-1)} = -\frac{24}{\sqrt{38}}\,\frac{C^{(0)} + C^{(2)}}{N-1}, \tag{A 4}$$

$$C^{(1)} = -\frac{8}{3\sqrt{38}}\,\frac{C^{(0)}}{N-1}, \tag{A 5}$$

$$C^{(3)} = -\frac{8}{3\sqrt{38}}\,\frac{C^{(2)}}{N-1}. \tag{A 6}$$

The rest of the harmonics are less singular at $\kappa = \pm k_m$ and do not take part in the dominant balance as $\kappa \to -k_m$.

Substituting (A 4)–(A 6) into (A 2) and (A 3) yields

$$C^{(0)} = \frac{4C^{(0)} + 2C^{(2)}}{(N-1)(N-2)}, \qquad C^{(2)} = \frac{4C^{(2)} + 2C^{(0)}}{(N-1)(N-2)}.$$

Consistency between these two relations requires that $C^{(0)} = \pm C^{(2)}$, and N is determined accordingly to be $N = 4$ if $C^{(0)} = C^{(2)}$ or $N = 3$ if $C^{(0)} = -C^{(2)}$. Out of these two possibilities, $N = 4$ results in stronger singularities and provides the dominant behaviour as $\kappa \to -k_m$ (assuming that $C^{(0)} = C^{(2)} \equiv C$ is not zero, of course). Hence, $N = 4$ and, from (A 4) and (A 6),

$$C^{(-1)} = -\frac{16}{\sqrt{38}}\,C, \qquad C^{(3)} = -\frac{8}{9\sqrt{38}}\,C.$$

The proposed expressions (A 1) then agree with (5.1).

To determine the constant C, we solve the equation system (4.8) for $\mathscr{A}_n(\kappa)$ ($n \geqslant 0$) by series expansions in the form

$$\mathscr{A}_0 = \sum_{p=2}^{\infty} b_{0,p}\,|\kappa|^{p-1}, \tag{A 7a}$$

$$\mathscr{A}_n = \begin{cases} \displaystyle\sum_{p=n}^{\infty} b_{n,p}\,\kappa^{p-1} & (\kappa > 0) \\ \displaystyle\sum_{p=n}^{\infty} \tilde{b}_{n,p}\,\kappa^{p-1} & (\kappa < 0) \end{cases} \qquad (n \geqslant 1), \tag{A 7b}$$

with $b_{1,1} = \tilde{b}_{1,1} = \dfrac{1}{2\sqrt{38}}$, $b_{0,2} = -\dfrac{6}{19}$, $\tilde{b}_{2,2} = -b_{2,2} = \dfrac{1}{57}$, $b_{1,2} = \tilde{b}_{1,2} = -\dfrac{187}{228\sqrt{19}}$, etc., according to (4.6). Upon substitution of (A 7) into (4.8), the coefficients $b_{n,p}$ and $\tilde{b}_{n,p}$ ($n \geqslant 0$, $p \geqslant n$) satisfy certain recurrence relations which are then solved numerically as described in Yang (1996). It turns out that $b_{0,2p+1} = 0$ and, consistent with (5.1), it is verified numerically that

$$b_{0,2p} \sim C\,\frac{\sqrt{2}}{3}\,2^{p+4}\,p(p+1)(p+2) \qquad (p \to \infty),$$

$$\tilde{b}_{2,p} \sim C\,\frac{(-\sqrt{2})^{p+1}}{3}\,p(p+1)(p+2) \qquad (p \to \infty),$$

where $C = -0.011$.

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